Asymmetric Buckling of Finitely Deformed Conical Shells

J. Famili*
The Babcock and Wilcox Company, Barberton, Ohio

The asymmetric buckling of truncated and complete conical shells under uniform hydrostatic pressure has been studied by many authors. In all of these investigations, membrane theory has been used to define the stress state prior to buckling. In the present paper this problem is studied by means of finite difference analysis, taking into account the large deformation in the prebuckling state. Numerical results are obtained and compared with the results of other investigations.

Introduction

The problem of the asymmetric buckling of conical shells has been the subject of a number of investigations in recent years. A reasonably complete bibliography of these papers is given in Ref. 1. In all of these investigations the linear membrane theory has been used to define the stress state prior to buckling. Since the buckling of the truncated conical shells is confined to the immediate vicinity of the small end, and since it has been shown in Refs. 2 and 3 that consideration of the bending effects are of importance near an edge to obtain more accurate results, it would seem to be essential to employ a complete theory that includes the bending effect.

The study of nonlinear axisymmetric buckling of a complete conical shell under uniform pressure was given by Newman and Reiss,[4] and the problem of asymmetric buckling of truncated and complete conical shells was presented by Niordson[5] and Seide.[6] Reiss' method of solution is essentially the same as was used in the earlier studies of the nonlinear axisymmetric bending of shallow spherical shells and with the same accuracy. Niordson's and Seide's solutions are based on the assumed linear membrane solutions prior to buckling and used the Rayleigh-Ritz method for the solution of the asymmetric buckling. Because of the assumed membrane-type solutions in the prebuckling state of stress, the accuracy of these solutions is in question, and so this area of stability theory remains to be explored.

In this paper, the numerical procedure developed in Refs. 7–9 and applied to shallow spherical shells is applied to the asymmetric buckling of complete and truncated conical shells. Numerical computations are carried out, leading to the determination of buckling loads for a range of shell geometries, and the results are compared with the results of other investigators.

Basic Equations

Consider a conical shell of thickness h and refer to the notations shown in Figs. 1 and 2. The basic equation suitable for stability analysis, in terms of displacement w, normal to the median surface and directed toward the axis of the cone and a stress function F, can be obtained from Ref. 10 if we replace $r = s$, $r_1 = \infty$, and $r_2 = s\tan\alpha$. Introducing the stress function F, which is related in the following manner to the stress resultants

$$N_s = (1/s)F_{,s} + 1/(s^2\sin^2\alpha)\,F_{,\theta\theta} \tag{1a}$$

$$N_\theta = F_{,ss} \tag{1b}$$

$$N_{s\theta} = \frac{1}{s^2\sin^2\alpha}F_{,\theta} - \frac{1}{s\sin\alpha}F_{,s\theta} = -\left(\frac{1}{s\sin\alpha}F_{,\theta}\right)_{,s} \tag{1c}$$

we get†

$$D\nabla^4 w = q_z + \left(\cot\alpha + w_{,s} + \frac{1}{s\sin^2\alpha}w_{,\theta\theta}\right)\frac{1}{s}F_{,ss} + \frac{2}{s^2\sin^2\alpha}\left[\frac{1}{s}w_{,\theta} - w_{,s\theta}\right]F_{,s\theta} + \frac{2}{s^2\sin^2\alpha}\times\left[w_{,\theta s} - \frac{1}{s}w_{,\theta}\right]\frac{1}{s}F_{,\theta} + w_{,ss}\frac{1}{s}F_{,s} + \frac{1}{s^2\sin^2\alpha}w_{,ss}F_{,\theta\theta} \tag{2a}$$

$$\frac{1}{Eh}\nabla^4 F = -\left(\cot\alpha + w_{,s} + \frac{1}{s\sin^2\alpha}w_{,\theta\theta}\right)\frac{1}{s}w_{,ss} - \frac{1}{s^2\sin^2\alpha}\left[\frac{2}{s}w_{,\theta} - w_{,s\theta}\right]w_{,s\theta} + \frac{1}{s^2\sin^2\alpha}\left(\frac{1}{s}w_{,\theta}\right)^2 \tag{2b}$$

where

$$D = Eh^3/[12(1-\nu^2)]$$

$$\nabla^2 = (\ )_{,ss} + \frac{1}{s}(\ )_{,s} + \frac{1}{s^2\sin^2\alpha}(\ )_{,\theta\theta}$$

and a comma, followed by subscripts, indicates differentiation with respect to subscripted variables. The strain displacement relations are shown in the form

$$\epsilon_s = u_{,s} + \tfrac{1}{2}(w_{,s})^2 \tag{3a}$$

$$\epsilon_\theta = \frac{1}{s}u + \frac{1}{s\sin\alpha}v_{,\theta} - \frac{\cot\alpha}{s}w + \frac{1}{2s^2\sin^2\alpha}(w_{,\theta})^2 \tag{3b}$$

$$\gamma_{s\theta} = \frac{1}{s\sin\alpha}u_{,\theta} - \frac{v}{s} + v_{,s} + \frac{1}{s\sin\alpha}w_{,s}w_{,\theta} \tag{3c}$$

$$\chi_s = -w_{,ss} \tag{3d}$$

$$\chi_\theta = -(1/s)w_{,s} - (1/s^2\sin^2\alpha)w_{,\theta\theta} \tag{3e}$$

$$\chi_{s\theta} = -2[(1/s\sin\alpha)w_{,s\theta} - (1/s^2\sin\alpha)w_{,\theta}] \tag{3f}$$

It should be noted that Eqs. (1) and (2) are identical to those given by Mushtari[11] and Schnell.[12]

Consider the following dimensionless variables

$$x = (s/s_1)\lambda \tag{4a}$$

$$\tilde{w} = w/s_1\cot\alpha \tag{4b}$$

$$\phi = (m^2/Eh^2)\cdot(F/s_1\cot\alpha) \tag{4c}$$

$$P = [(m^4 s_1^3)/(2h^3\cot\alpha)]\cdot(q_z/E) \tag{4d}$$

$$\lambda^2 = m^2 s_1\cot\alpha/h \tag{4e}$$

$$m^4 = 12(1-\nu^2) \tag{4f}$$

$$\bar{\theta} = \theta\sin\alpha \tag{4g}$$

Received November 16, 1964; revision received March 31, 1965. The numerical computations reported in this work were carried out on the Univac 1107 Computer at the Case Institute of Technology, while the author was a Research Assistant in the Engineering Division. It was supported, in part, by National Science Foundation Grant No. GP-938. The author wishes to express his thanks to Robert Archer for his interest and many valuable suggestions during the course of this research.

* Design Engineer.

† Our primary interest here is in the buckling problem, so that the inertia terms are not included.

With the aid of Eqs. (4), the nondimensional form of Eqs. (2) can be obtained as

$$\bar{\nabla}^4\tilde{w} = \frac{2P}{\lambda^4} + \left(\frac{1}{\lambda x} + \frac{1}{x}\tilde{w}_{,x} + \frac{1}{x^2}\tilde{w}_{,\bar{\theta}\bar{\theta}}\right)\lambda^2\phi_{,xx} + \frac{2\lambda^2}{x^2}\left(\frac{1}{x}\tilde{w}_{,\bar{\theta}} - \tilde{w}_{,\bar{\theta}x}\right)\phi_{,\theta x} + \frac{2\lambda^2}{x^3}\left(\tilde{w}_{,\bar{\theta}x} - \frac{1}{x}\tilde{w}_{,\bar{\theta}}\right)\phi_{,\bar{\theta}} + \frac{\lambda^2}{x}\phi_{,x}\tilde{w}_{,xx} + \frac{\lambda^2}{x^2}\tilde{w}_{,xx}\phi_{,\bar{\theta}\bar{\theta}} \quad (5a)$$

$$\bar{\nabla}^4\phi = -\left(\frac{1}{\lambda x} + \frac{1}{x}\tilde{w}_{,x} + \frac{1}{x^2}\tilde{w}_{,\bar{\theta}\bar{\theta}}\right)\lambda^2\tilde{w}_{,xx} + \frac{\lambda^2}{x^2}\left(\frac{1}{x}\tilde{w}_{,\bar{\theta}} - \tilde{w}_{,\theta x}\right)^2 \quad (5b)$$

where

$$\bar{\nabla}^2 = (\ )_{,xx} + \frac{1}{x}(\ )_{,x} + \frac{1}{x^2}(\ )_{,\bar{\theta}\theta}$$

Equations (5) are basic equations for the study of the stability problem in this work

Axisymmetric Buckling of Conical Shells

The axisymmetric nonlinear primary state of stress and deflection can be obtained from (5) by dropping all of the terms involving derivatives with respect to θ. Using a stress function and a slope variable, these equations are of the form

$$x^2f'' + xf' - f - x\lambda g = (Px^3/\lambda^4) + x\lambda^2 fg \quad (6a)$$

$$x^2g'' + xg' - g + x\lambda f = -\tfrac{1}{2}x\lambda^2 f^2 \quad (6b)$$

where $(\ )' = d/dx(\ )$

$$f = \tilde{w}' \equiv (\tan\alpha/\lambda)(dw/ds) \quad (7a)$$

$$g = \phi' \equiv [(sm^2)/(Eh^2\lambda\cot\alpha)](N_s)_0 \quad (7b)$$

Equations (6) are identical to Reiss' equations[4] for nonlinear axisymmetric buckling of complete conical shells if one replaces

$$f = \frac{y}{\lambda} \qquad g = \frac{z}{\lambda^2} \qquad x = x_r\lambda \qquad \lambda^2 = k$$

where y, z, and x_r are in the Reiss notation. As explained in Refs. 8 and 9, it is to be expected that prebuckling deformations are axisymmetric and may be obtained directly from Eqs. (6) with appropriate boundary conditions.

If we define $\tilde{w}_0(x)$ and $\phi_0(x)$ to be the axisymmetric solution and write

$$\tilde{w} = \tilde{w}_0(x) + \tilde{w}(x, \bar{\theta}) \quad (8a)$$

$$\phi = \phi_0(x) + \bar{\phi}(x, \bar{\theta}) \quad (8b)$$

then, substituting (8) into the Eqs. (5), using (7), and retaining only linear terms in $\tilde{w}$ and $\bar{\phi}$, we get

$$\bar{\nabla}^4\tilde{w} = \left(\frac{1}{\lambda x} + \frac{1}{x}f_0\right)\lambda^2\bar{\phi}_{,xx} + \left(\frac{1}{\lambda}\tilde{w}_{,x} + \frac{1}{x^2}\tilde{w}_{,\bar{\theta}\bar{\theta}}\right)\lambda^2 g_0' + \left(\frac{1}{x}\bar{\phi}_{,x} + \frac{1}{x^2}\bar{\phi}_{,\bar{\theta}\bar{\theta}}\right)\lambda^2 f_0' + \frac{\lambda^2}{x}\tilde{w}_{,xx}g_0 \quad (9a)$$

$$\bar{\nabla}^4\bar{\phi} = -\left(\frac{1}{\lambda x} + \frac{1}{x}f_0\right)\lambda^2\tilde{w}_{,xx} - \left(\frac{1}{x}\tilde{w}_{,x} + \frac{1}{x^2}\tilde{w}_{,\bar{\theta}\bar{\theta}}\right)\lambda^2 f_0' \quad (9b)$$

The displacement $\tilde{w}(x, \bar{\theta})$ and stress function $\bar{\phi}(x, \bar{\theta})$ satisfies the boundary conditions consistent with the axisymmetric solution. Equations (9) are a system of eighth order linear partial differential equations that may be converted to ordinary differential equations relating $\tilde{w}$, $\bar{\phi}$, and the eigenvalue p, by assuming a solution of the type

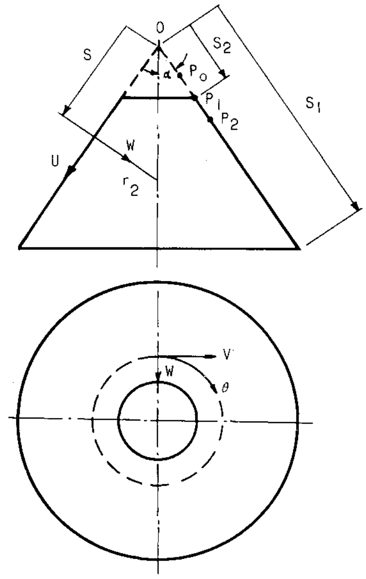

Fig. 1 Middle surface of a conical shell.

$$\tilde{w} = \tilde{w}_0(x) + \tilde{w}_n(x)\cos\bar{n}\bar{\theta} \quad (10a)$$

$$\bar{\phi} = \phi_0(x) + \bar{\phi}_n(x)\cos\bar{n}\bar{\theta} \quad (10b)$$

where $\bar{n} = n/\sin\alpha$ (n is the wave number defining the number of circumferential buckles in the cone). This type of solution has already been used in Refs. 17 and 18 for the calculation of buckling loads in cylindrical and shallow spherical shells.

Boundary Conditions

In order to compare the numerical results with Reiss[4] in the range of axisymmetric bending, and with Niordson[5] and Seide[6] in the asymmetric range, we used Seide's boundary conditions. He assumed that the shell is closed by hypothetical bulkheads that are rigid in their plane, but free to distort out of plane, i.e.,

$$u\sin\alpha - w\cos\alpha = 0 \quad (11a)$$

$$v = 0 \quad (11b)$$

$$M_s = 0 \quad (11c)$$

$$(\cot\alpha + w_{,s})sN_s + (sM_s)_{,s} - M_\theta + 2(M_{s\theta})_{,\theta} = 0 \quad (11d)$$

The last equation is a natural boundary condition and is obtained directly from an energy expression.[6] Using a stress-strain relation, replacing $M_s + (M_s)_0$, $N_s + (N_s)_0$, and $w + w_0$ for M_s, N_s, and w, respectively, and retaining linear terms only, then the nondimensional form of the boundary conditions

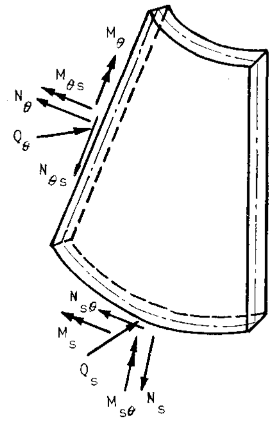

Fig. 2 Force and moment resultants for conical shell.

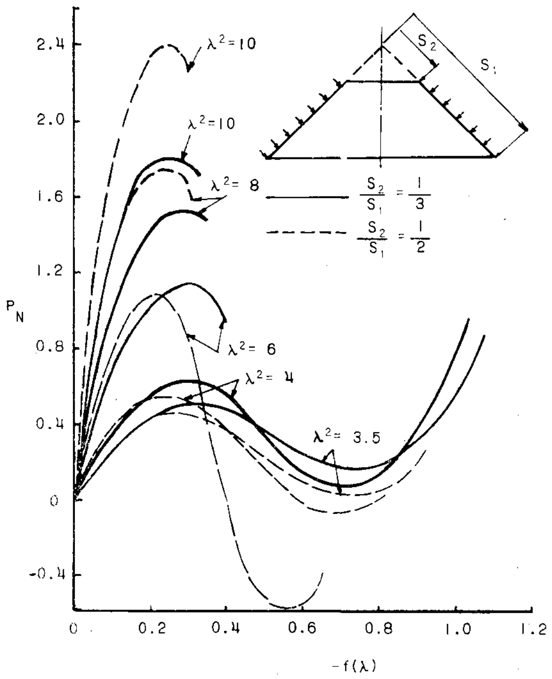

Fig. 3 Base slope vs load for axisymmetric truncated conical shell.

in terms of w and ϕ can be obtained as follows:

$$\bar{\phi}_{,xx} - (\nu/x)[\bar{\phi}_{,x} + (1/x)\bar{\phi}_{,\theta\theta}] = 0 \qquad (12a)$$

$$x\left(\bar{\phi}_{,xx} - \frac{\nu}{x}\bar{\phi}_{,x} - \frac{\nu}{x^2}\bar{\phi}_{,\theta\theta}\right)_{,x} - \frac{1}{x^3}\bar{\phi}_{,x}\bar{\phi}_{,\theta\theta} + \nu\bar{\phi}_{,xx} +$$
$$2(1+\nu)\left(\frac{1}{x}\bar{\phi}_{,\theta\theta}\right)_{,x} - \frac{\lambda}{\nu}(1+\lambda f_0)x\bar{w}_{,xx} = 0 \qquad (12b)$$

$$\bar{w}_{,xx} + (\nu/x)[\bar{w}_{,x} + (1/x)\bar{w}_{,\theta\theta}] = 0 \qquad (12c)$$

$$(1+\lambda f_0)\bar{\phi}_{,xx} - \frac{\nu}{\lambda}\bar{w}_{,xxx} - \frac{(1+\nu)}{x\lambda}\bar{w}_{,xx} +$$
$$\frac{\nu(\nu+2)}{\lambda x^3}(x\bar{w}_{,\theta\theta x} - \bar{w}_{,\theta\theta}) + \frac{\nu\lambda}{x}g_0\bar{w}_{,x} = 0 \qquad (12d)$$

at

$$x = (s_2/s_1)\lambda \quad \text{and} \quad x = \lambda$$

For the axisymmetric solution (6), the corresponding boundary conditions are at $x = (s_2/s_1)\lambda$, $x = \lambda$

$$\bar{\phi}'' - (\nu/x)\bar{\phi}' = 0 \qquad (13a)$$

$$\bar{w}'' + (\nu/x)\bar{w}' = 0 \qquad (13b)$$

i.e., the meridional bending moment and horizontal displacement vanish.

The boundary conditions (12) and (13) can also be applied at the open end of a complete conical shell, but in a sufficiently small neighborhood of the apex, since the shell is not "thin," the Eqs. (2) may be invalid. To circumvent this difficulty, we assume a spherical inclusion near the apex, which is practical in shell fabrication. Therefore, for the asymmetric buckling problem (complete conical shell only), we may assume that near $x = 0$, $\bar{w} \sim x^n$, and $\bar{\phi} \sim x^n$. For the axisymmetric buckling problem, we have, in any case, $f(0) = g(0) = 0$.

Buckling of Truncated Conical Shells

Method of Solution

The numerical method for the buckling analysis of conical shells is similar to that previously employed in Refs. 8 and 9 for shallow spherical shells. We start the solution of axisymmetric nonlinear differential Eqs. (6). In order to speed up the rate of convergence, the nonlinear terms in Eqs. (6) are replaced by series expansions about the kth iterate terms in powers of the difference between k + 1st and kth iterate. Retaining only first-order terms, it follows

$$f^{k+1}g^{k+1} = f^kg^k + f^k(g^{k+1} - g^k) + g^k(f^{k+1} - f^k) \qquad (14)$$

Replacing all derivatives by central differences and using a block tridiagonal scheme, the Eqs. (6) can be written as

$$A_iT_{i-1} + B_iT_i + C_iT_{i+1} = D_i \qquad (15)$$

where

$$A_i = \begin{bmatrix} (x^2/\delta^2) - (x/2\delta) & 0 \\ 0 & (x^2/\delta^2) - (x/2\delta) \end{bmatrix}$$

$$B_i = \begin{bmatrix} -1 - (2x^2/\delta^2) - x\lambda^2g_i & -x\lambda - x\lambda^2f_i \\ x\lambda + x\lambda^2f_i & -1 - (2x^2/\delta^2) \end{bmatrix}$$

$$C_i = \begin{bmatrix} (x^2/\delta^2) + (x/2\delta) & 0 \\ 0 & (x^2/\delta^2) + (x/2\delta) \end{bmatrix} \quad T_i = \begin{bmatrix} f \\ g \end{bmatrix}_i$$

$$D_i = \begin{bmatrix} (px^3/\lambda^4) - x\lambda^2f_ig_i \\ \frac{1}{2}x\lambda^2f_i^2 \end{bmatrix} \qquad i = 2, 3, \ldots, N-1$$

where δ is the uniform spacing parameter.

Axisymmetric Boundary Conditions

The boundary conditions (13) can be replaced by the vector conditions

$$B_1T_1 + C_1T_2 = D_1 \qquad (16a)$$

$$A_NT_{N-1} + B_NT_N = D_N \qquad (16b)$$

Consider a generator of the cone Fig. 1, using the axisymmetric Eqs. (6) and boundary conditions (13) in vector form at point P_1 and introducing one fictitious station $i = 0$ off the inside edge, we can write

$$A_1T_0 + B_1T_1 + C_1T_2 = D_1 \qquad (17)$$

$$A_0T_0 + B_0T_1 + C_0T_2 = 0 \qquad (18)$$

Eliminating T_0

$$(A_1C_0^{-1}B_0 + B_1)T_1 + (A_1 + C_1)T_2 = D_1 \qquad (19)$$

where

$$(A_1C_0^{-1}B_0 + B_1) =$$

$$\begin{bmatrix} 2[\frac{\nu\delta}{\mu\lambda} - 1]\frac{\mu\lambda}{\delta}{}^2 - 1 - \nu - \mu\lambda^3g_1 & -\mu\lambda^2 - \mu\lambda^3f_1 \\ \mu\lambda^2 + \mu\lambda^3f_1 & -2[\frac{\nu\delta}{\mu\lambda} + 1]\frac{\mu\lambda}{\delta}{}^2 - 1 + \nu \end{bmatrix}$$

$$(A_1 + C_1) = \begin{bmatrix} 2(\mu\lambda/\delta)^2 & 0 \\ 0 & 2(\mu\lambda/\delta)^2 \end{bmatrix}$$

$$A_N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B_N = \begin{bmatrix} 1 + (\delta\nu/\lambda) & 0 \\ 0 & 1 - (\delta\nu/\lambda) \end{bmatrix}$$

$$\mu = s_1/s_2$$

The differential equation (15) and boundary conditions (16) can be solved for f, g, f_i', and g_i' by using block tridiagonal iterative scheme.‡

Using Eqs. (10) and axisymmetric solution, the differential equations (9) and boundary conditions (12) for asymmetric buckling of conical shells reduce to the following form

$$\bar{\xi}_n'' + \frac{2}{x}\bar{\xi}_n{}^1 - \frac{1+2\bar{n}^2}{x^2}\bar{\xi}_n + \frac{1+2\bar{n}^2}{x^3}\bar{w}_n' -$$
$$\frac{4\bar{n}^2 - \bar{n}^4}{x^4}\bar{w}_n = \left(\frac{1}{x\lambda} + \frac{1}{x}f_0\right)\lambda^2\bar{\eta}_n + \left(\frac{1}{x}\bar{w}_n' - \frac{\bar{n}^2}{x^2}\bar{w}_n\right) \times$$
$$\lambda^2g_0' + \left(\frac{1}{x}\bar{\phi}_n' - \frac{\bar{n}^2}{x^2}\bar{\phi}_n\right)\lambda^2f_0' + \frac{\lambda^2}{x}\bar{\xi}_ng_0 \qquad (20a)$$

‡ See Ref. 7 for this iterative method.

$$\bar{\eta}_n'' + \frac{2}{x}\bar{\eta}_n' - \frac{1+2\bar{n}^2}{x^2}\bar{\eta}_n + \frac{1+2\bar{n}^2}{x^3}\bar{\phi}_n' - \frac{4\bar{n}^2 - \bar{n}^4}{x^4}\bar{\phi}_n = -\left(\frac{1}{x\lambda} + \frac{1}{x}f_0\right)\lambda^2\bar{\xi}_n - \left(\frac{1}{x}\bar{w}_n' - \frac{\bar{n}^2}{x^2}\bar{w}_n\right)\lambda^2 f_0' \quad (20b)$$

where

$$\bar{\xi}_n = \bar{w}_n'' \quad (20c)$$

$$\bar{\eta}_n = \bar{\phi}_n'' \quad (20d)$$

Boundary conditions at $x = (s_2/s_1)\lambda$ and $x = \lambda$ are

$$\bar{\xi}_n + (\nu/\lambda)\bar{w}_n' - (\bar{n}^2\nu/\lambda^2)\bar{w}_n = 0 \quad (21a)$$

$$(1+\lambda f_0)\bar{\xi}_n - \frac{\nu}{\lambda}\bar{\eta}_n' + \frac{\nu}{\lambda^3}[(1-\nu) + (2+\nu)\bar{n}^2]\bar{\phi}_n' - \frac{3\nu\bar{n}^2}{\lambda^4}\bar{\phi}_n = 0 \quad (21b)$$

$$\bar{\eta}_n - (\nu/\lambda)\bar{\phi}_n' + (\bar{n}^2\nu/\lambda^2)\bar{\phi}_n = 0 \quad (21c)$$

$$(1+\lambda f_0)\bar{\eta}_n - \frac{\nu}{\lambda}\bar{\xi}_n' - \frac{(1+\nu)}{\lambda^2}\bar{\xi}_n + \left[\frac{\nu(2-\nu)\bar{n}}{\lambda^3} + \nu g_0\right]\bar{w}_n' - \frac{\nu(2-\nu)}{\lambda^4}\bar{w}_n = 0 \quad (21d)$$

Replacing all first and second derivatives by central differences, then the eighth order system of differential equations (20) reduce to a block tridiagonal matrix of the form

$$A_i\bar{T}_{i+1} + B_i\bar{T}_i + C_i\bar{T}_{i+1} = 0 \quad (22)$$

where

$$\bar{T}_i = \begin{bmatrix}\bar{w}_n \\ \bar{\phi}_n \\ \bar{\xi}_n \\ \bar{\eta}_n\end{bmatrix}_i \quad i = 2, 3, \ldots N$$

and A_i, B_i, C_i are 4×4 matrices.§ By introducing one fictitious station $i = N + 1$ off the large end of the shell for convenience, we get two vector equations

$$A_0\bar{T}_0 + B_0\bar{T}_1 + C_0\bar{T}_2 = 0 \quad (23a)$$

$$A_{N+1}\bar{T}_{N-1} + B_{N+1}\bar{T}_N + C_{N+1}\bar{T}_{N+1} = 0 \quad (23b)$$

The elements of matrices A_0, B_0, C_0, A_{N+1}, B_{N+1}, and C_{N+1} depend on the boundary conditions at the small and large ends of the shell.

Again, writing the differential equation (22) at point P_1 (Fig. 1) and eliminating T_0 between this equation and (23a), one can get

$$(B_1 - A_1A_0^{-1}B_0)\bar{T}_1 + (C_1 - A_1A_0^{-1}C_0)\bar{T}_2 = 0 \quad (23c)$$

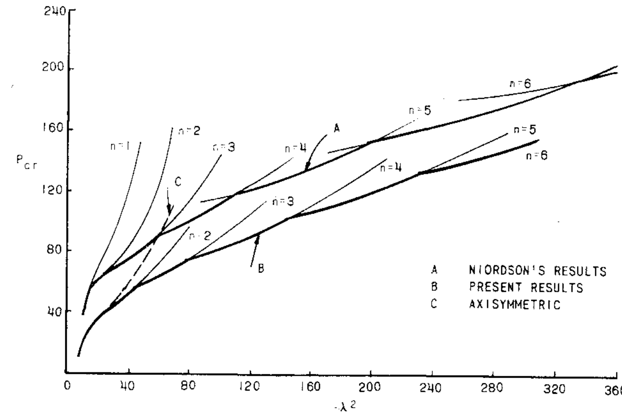

Fig. 4 Asymmetric buckling pressures of truncated conical shell.

We can now perform the same block tridiagonal reduction scheme that was used in Refs. 8 and 9 to solve the buckling problems of shallow spherical shells

$$\bar{T}_i + R_i\bar{T}_{i+1} = 0 \quad i = 1, 2, \ldots N \quad (24a)$$

$$W\bar{T}_{N+1} = 0 \quad (24b)$$

where

$$R_i = (B_i - A_iR_{i-1})^{-1}C_i \quad (24c)$$

$$W = C_{N+1} - (B_{N+1} - A_{N+1}R_{N-1})R_N \quad (24d)$$

Thus, for a nontrivial solution $\bar{T}_{N+1}$, we must have

$$|W| = 0 \quad (25)$$

Therefore, the value of p can be obtained such that Eq. (25) is satisfied.

Numerical Results and Conclusion

Numerical results were obtained for a shell problem with constant shell geometry parameters $\alpha = 45°$, $\mu = 1/2$, $\nu = 0.3$, and $3.5 \leq \lambda^2 \leq 300$. To check the computer program, the numerical procedure was applied to a specific problem, the "Belleville Spring" (shallow truncated conical shell) subjected to compressive axisymmetric axial loads with the assumption that the edges are free to rotate and move radially. The numerical results are compared against Reiss,[4] and the experimental results of Almen and Laszlo.[14] Excellent agreement between this solution and Ref. 4 was obtained. Corresponding numerical values and graphs are shown in Ref. 13.

To check the general form of the computer program, the problem of asymmetric buckling of shallow spherical shells was solved by using a displacement w and stress function F formulation. It was found that the results are the same as those obtained in Refs. 8 and 18. Since the modifications

Table 1 Load and wave number with different values of λ^2 for asymmetric truncated cone

$n = 2^a$		$n = 3$		$n = 4$		$n = 5$		$n = 6$	
λ_2	P	λ^2	P	λ^2	P	λ^2	P	λ^2	P
34.64	48.885	34.64	53.560	86.60	76.643	155.88	104.556	207.85	126.608
43.30	55.979	51.96	59.372	103.92	82.504	173.20	109.645	225.17	130.541
51.96	64.307	69.28	67.699	121.24	89.752	190.53	115.505	242.49	134.319
60.62	74.176	86.60	79.419	138.56	98.542	207.85	121.982	259.81	138.946
69.28	85.433	103.92	94.069	155.88	108.257	225.17	129.076	277.13	143.803
...	...	121.24	111.496	173.20	119.360	242.49	136.632	294.45	147.427
...	...	...	...	190.53	131.852	259.81	144.806	...	...

[a] Note that for truncated conical shells, evaluation of buckling load for $n = 1$ does not create any problem. But, for complete conical shells, one has to write a special form of Eq. (23c), such as given in Ref. 8 for shallow spherical shells. Since the computer program has been assigned for both truncated and complete cones, no attempt has been made to compute buckling load for $n = 1$.

§ The elements of these matrices are given in Ref. 13.

Table 2 Load and wave numbers with different values of λ^2 for asymmetric complete cone

$n = 2$		$n = 3$		$n = 4$		$n = 5$	
λ^2	P	λ^2	P	λ^2	P	λ^2	P
25.98	34.852	34.64	49.193	60.62	69.550	121.243	95.766
34.64	39.562	43.30	51.661	77.94	72.634	138.564	99.004
43.30	46.110	51.96	54.051	95.263	76.489	155.884	102.243
51.96	53.743	60.62	57.058	103.923	78.879	173.205	106.524
60.62	62.456	77.94	64.152	121.244	84.050	190.530	110.960
...	...	103.923	78.494	138.546	90.137	207.846	115.814
...	...	121.244	89.675	155.885	96.845	225.167	120.830
...	...	...	...	173.205	103.708	242.487	125.992
...	...	...	...	...	...	259.807	131.466

in going from the spherical geometry to the conical are a minor part of the program, the results were a good indication that the program was accurately constructed.

Since there appear to be no numerical results published for the axisymmetric nonlinear truncated cone, this problem was solved separately for a number of shell geometries and the results are shown in Fig. 3.

For the study of asymmetric buckling, a stability determinate is obtained as a function of λ, n, and P. The zeros of this determinate reveal the buckling loads for the given geometry λ^2 and circumferential mode type n. The values of the buckling load for different values of shell geometry were obtained and the results were compared with Niordson's[5] solution. Corresponding graphs are shown in Fig. 4 and the numerical values are given in Table 1.

Also, the computations were made for complete conical shells (shell closed at one end) for different values of shell geometry λ^2, both for axisymmetric primary state alone and for asymmetric buckling in the neighborhood of the primary state. The axisymmetric results are compared with Ref. 4. It appears that the results of this work and Ref. 4 are in good agreement. For the asymmetric buckling, the results are compared with Ref. 5 approximate theory for $3.5 \leq \lambda^2 \leq 260$ and are shown graphically in Fig. 5, and numerical values are given in Table 2.

The results of this work show that the buckling loads for both truncated and complete conical shells are lower than those computed from Niordson's theory, and it seems to be reasonable since Niordson's solution is based on the Ritz type energy method and should be expected to indicate large buckling loads. Seide's[6] solution is also based on the modified Ritz method; however, his results are higher than Niordson's.

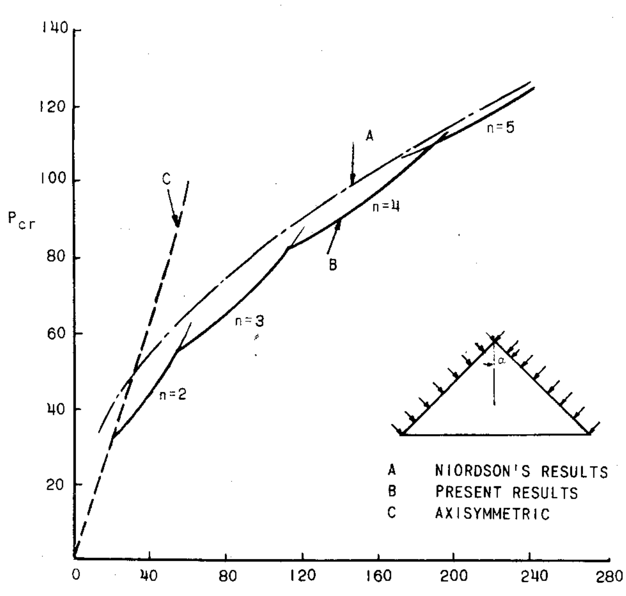

Fig. 5 Buckling pressure for a complete cone.

Some authors believe that the discrepancy between Niordson's and Seide's solutions was due to some simplifications that Niordson made in the final stages of his Ritz solution. To compare the results of this work with Niordson's, for truncated conical shells, we neglected his simplifying assumptions and coded his complete equations for the computer. The buckling loads for different values of shell geometry and wave numbers are obtained and the results were compared with this present work.

As noted before, for complete conical shells we used Niordson's approximate theory and it seems that as λ^2 becomes larger and larger, his approximate solution for buckling loads is closer to the present solution. Bijlaard[15] also studied the buckling of complete cones as well as truncated conical shells. His buckling loads for complete conical shells are lower than the results of this work, as well as the experimental results.[16] The experimental results are scattered between Niordson's results as upper bound and Bijlaard's as lower bound. The results of this work are between Niordson's and Bijlaard's results. Unfortunately, there are no experimental results available for this problem over the range of $\lambda^2 < 700$. In order to obtain some results for the $\lambda^2 < 700$ by the finite difference method, the number of mesh points N must be very large. Considerable computer time would be required. The numerical values reported in Ref. 6 are for very thin shells that correspond to large values of λ in the present paper ($\lambda^2 > 2000$); therefore, no attempt has been made to compare the present results with Ref. 6. Apparently, the buckling load reaches an asymptotic value as λ^2 is increased in the limit, and the effect of the bending stress becomes less important except for a boundary layer at the edge, where large bending stresses also occur.

References

[1] Seide, P., "A survey of buckling theory and experiment for circular conical shells of constant thickness," NASA TN D-1510, pp. 401–426 (1962).

[2] Thurston, G. A., "A numerical solution for thin conical shells under asymmetrical loads," *Proceedings of the 4th Midwestern Conference on Solid Mechanics* (The University of Texas, Austin, Texas, 1959), pp. 171–194.

[3] Clark, R. A. and Garibotti, J. F., "Longitudinal bending of a conical shell," Douglas Missile and Space Systems Div., Engineering Paper 1547 (March 7, 1963).

[4] Newman, M. and Reiss, E. L., "Axisymmetric snap buckling of conical shells," NASA TN D-1510, pp. 451–462 (1962).

[5] Niordson, F. I. N., "Buckling of conical shells subjected to uniform external lateral pressure," Trans. Roy. Inst. of Technol., Stockholm **10,** 1–12 (1947).

[6] Seide, P., "On the buckling of truncated conical shells under uniform hydrostatic pressure," *Proceedings of the I. V. T. A. M. Symposium on Thin Elastic Shells, Delft, 24–25 August,* 1959 (North-Holland Publishing Co., Amsterdam, 1960), pp. 363–386.

[7] Archer, R. R., "On the numerical solution of the non-linear equations for shells of revolution," J. Math. Phys. **41,** 165–178 (1962).

[8] Archer, R. R., and Famili, J., "On the vibration and stability of finitely deformed shallow spherical shells," Appl. Mech. **32,** 116–120 (1965).

[9] Famili, J. and Archer, R. R., "On the finite asymmetric deformation of shallow spherical shells," AIAA J. **3**, 506–510 (1965).
[10] Langhaar H., and Boresi, A., "Buckling and post buckling of elastic shells," NASA TN D-1510, pp. 115–133 (1962).
[11] Mushtari, K. M. and Galimore, K. Z., *Non-Linear Theory of Thin Elastic Shells*, transl. from Russian by Kajan (U. S. Department of Commerce, Washington D. C., 1957).
[12] Schnell, W., "Die Dunnwandige Kegelschale unter Axial und Innendruck," Z. Flugwiss **10**, 314, 321 (1962).
[13] Famili, J., "On the vibration, buckling and post-buckling behavior of shells of revolution under symmetrical and unsymmetrical load," Ph.D. Thesis, Case Institute of Technology (1964).
[14] Almen, J. O. and Laszlo, A., "The uniform-section disk spring," Trans. Am. Soc. Mech. Engrs. **58**, 305–314 (1936).
[15] Bijlaard, P. P., "Buckling of thin conical shells under uniform external pressure," NASA TN, D-1510, pp. 441–450 (1962).
[16] Jordan, W. D., "Buckling of thin conical shells under uniform external pressure," Technical Report, Bureau of Engineering Research, College of Engineering, Univ. of Alabama (February 1955).
[17] Stein, M., "The effect on the buckling of perfect cylinders of pre-buckling deformations and stresses induced by edge support," NASA TN D-1510, pp. 217–227 (1962).
[18] Huang, N. C., "Unsymmetrical buckling of thin shallow spherical shells," J. Appl. Mech. **31**, 447–457 (1964).

AUGUST 1965 AIAA JOURNAL VOL. 3, NO. 8

Axial Buckling of Pressurized Imperfect Cylindrical Shells

John Hutchinson*
Harvard University, Cambridge, Mass.

Whereas some experiments seem to confirm the contention that the axial buckling load of a pressurized cylinder approaches the classical value (i.e., the value predicted by the linear buckling equations) for sufficiently large internal pressure, other tests indicate a much smaller buckling load increase resulting from pressurization. Here it is shown that the buckling load of an elastic shell with asymmetric imperfections, but sufficiently free of axisymmetric imperfections, closely coincides with the classical value for relatively small values of internal pressure. However, the buckling load of a shell with a predominance of axisymmetric imperfection can remain well below the classical value for the entire range of internal pressures for which the shell buckles elastically. Of particular interest is the calculation of an upperbound to the buckling load as predicted by the nonlinear Donnell-shell equations for a shell with axisymmetric imperfections.

Introduction

THE several published analyses of pressurized cylindrical shells under axial compression do not adequately explain the variety of behavior that has been reported for such structures. The linear buckling equations predict that the buckling parameter

$$\lambda = [3(1 - \nu^2)]^{1/2}(R/Eh)[\sigma - (pR/2h)]$$

is unity for pressurized and unpressurized shells. Here σ is the applied compressive stress, and $pR/2h$ is the axial stress resulting from pressurization as depicted in Fig. 1. Experiments indicate that this buckling parameter is usually on the order of one-half or one-third for unpressurized shells and is larger for pressurized shells. In some tests the parameter is unity for shells under sufficient internal pressure, although in other cases this parameter remains well below unity for the entire range of pressures for which the shell buckles elastically.

Lo, Crate, and Schwartz[1] used nonlinear buckling equations for a perfect shell and employed Tsien's energy criterion of buckling to show that the buckling parameter, as defined here, increases from 0.62 at $p = 0$ to unity at $pR^2/Eh^2 = 0.17$. Thieleman[2] also reported calculations for initially perfect shells based on a nonlinear analysis. He obtained load-deflection curves from which he determined the minimum load that the shell can support following buckling. This minimum load increases with increased internal pressure.

The role of shell imperfections, known to be the main degrading factor in unpressurized shells, was not considered in either of the previously mentioned papers. Lu and Nash[3] have studied the effect of initial imperfections on the minimum load that the shell can support in the postbuckling region. This analysis also shows a larger minimum load for pressurized shells than unpressurized ones.

From a design standpoint, the maximum load that the shell can support prior to buckling is of more interest than the

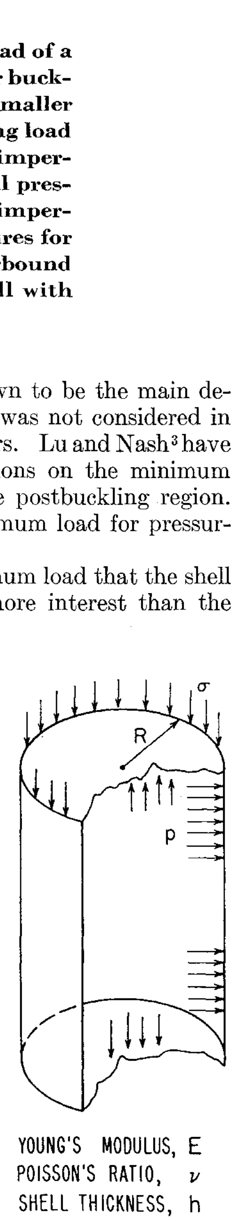

Fig. 1 Shell configuration.

Received December 2, 1964; revision received April 2, 1965. This work was supported by NASA under Contract No. NsG-559 with Harvard University.

* Assistant Professor of Structural Mechanics.